Final Exam Mathematical Physics, Prof. G. Palasantzas

- Date 17-06-2016
- Total number of points 100
- 10 points free for coming to the final exam
- For all problems justify your answer

<u>Problem 1</u> (10 points)

Show that $$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$ with $x \in (-\infty, +\infty)$

Problem 2 (15 points)

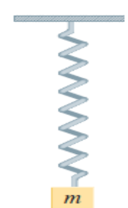
Consider the power series: $$\sum_{n=1}^{\infty} \frac{n}{3^n}(x-5)^n$$

Find the range of convergence of the given series (*Tip: Find the values of x for which the series is absolutely and conditionally convergent*)

Problem 3 (20 points)

Suppose a mass m is attached to a spring with spring constant k so that $k = m\omega^2$. If an external force $F(t) = F_o \cos(\omega t)$ is applied to the mass m, then the equation of motion of the mass m will be given by:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \quad \text{with } c^2 - 4mk < 0$$



Then prove* that the motion is described by the general solution :

$$x(t) = e^{-(c/2m)t}\left[c_1\cos(\tilde{\omega}t) + c_2\sin(\tilde{\omega}t)\right] + \left(\frac{F_o}{c\omega}\right)\sin(\omega t)$$

with $\tilde{\omega} = \omega\sqrt{1-(c/2m\omega)^2}$ and c_1, c_2 constants .

<u>It is not a proof</u> to substitute the solution and show that it satisfies the equation of motion. You have to derive the general solution using the method of undetermined coefficients.

Problem 4 (10 points)

Find the periodic solutions in complex form of the equation $\frac{d^2W}{dx^2} + AW = f(x)$

with A ($\neq$0) a real (non-integer) number, and f (x) a known 2π-periodic function.

To this end consider the Fourier series form for f and W: $f(x) = \sum_{n=-\infty}^{n=+\infty} f_n e^{inx}$, $W(x) = \sum_{n=-\infty}^{n=+\infty} W_n e^{inx}$

Problem 5 (15 points)

Assume a function *f(x)* has the Fourier transform: $F(k) = \int_{-\infty}^{+\infty} f(x) e^{-i2\pi kx} dx$

Consider the definition of the Dirac delta function: $\delta(k) = \int_{-\infty}^{+\infty} e^{-i2\pi kx} dx$

(a: 5 points) Derive the Fourier transform of the function: $f(x) = \sin[2\pi k_o x]$

(b: 10 points) Derive the Fourier Transform of the function: $f(x) = \sin^3[2\pi k_o x]$

Tip: sin(x)=(e^{ix} *-* e^{-ix} *)/2i, and if it helps consider the identity sin(3x)=3sin(x)-4*$\sin^3$*(x)*

Problem 6 (20 points)

Consider the boundary value problem for the one-dimensional heat equation for a bar with the zero temperature ends, where u(x,t) is the temperature:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad t > 0, \quad 0 < x < L,$$
$$u(x,0) = f(x), \quad (1)$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0.$$

(a: 10 points) Prove that the general solution for the boundary value problem (1) is given by: $u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x$

(b: 5 points) Derive the solution *u(x,t)* for the case *f(x)=Asin(mπx/L)* where m=9 and A a real positive number

(c: 5 points) Derive the solution *u(x,t)* for the case $f(x)=A_1\sin(m_1\pi x/L)+A_2\sin(m_2\pi x/L)+A_3\sin(m_3\pi x/L)$ where $(m_1, m_2, m_3) = (10, 20, 30)$ and $A_{j=(1,2,3)}$ are real positive numbers.